Lienhard Wimmer
mathematics@lienhard-wimmer.com
lienhard.wimmer@web.de

Answer to a Question of Marek Teichmann

Marek Teichmann once asked Dave Rusin per e-mail:
(see *http://www.math.niu.edu/ rusin/known-math/index/spheres.html*)

Problem 1 *Place N points on a unit sphere in R^3, so that the largest sphere concentric with the first, and inscribed in the convex hull of the points, is as large as possible.*

What is a lower bound on the radius of the inscribed sphere given N?

In other words, if the sphere is centered at the origin, we wish to maximize the distance from the origin to the closest facet of the convex hull of the points.

This question is equivalent to the following

Problem 2 *How must a sphere be covered by n equal circles where every point of the sphere belongs to at least one circle so that the angular radius of the circles will be as small as possible ?*

Because the second problem is quite well known and had been revisited by several authors (see the bibliography below). Therefore for many values of N numerical results are aviable and the exact solution is known for:

N = 2, 3, 4, 12	Lászlo Fejes-Tóth [2]
N = 5, 6, 7	Kurt Schütte [3]
N = 8, 9	Lienhard Wimmer [5]
N = 10, 14	Gabor Fejes-Tóth [1]

Proof, that the problems are equivalent:
We may assume without loss of generality that the radius of the sphere is equal to 1 and that the convex hull of the points is triangulated by diagonals. Then let ABC be a triangle of the convex hull. Obvisousely the perpendical point of the sphere's origin O onto ABC is the circumcenter U of ABC and the distance OU is given by $\sqrt{1-R^2}$, if R denotes the plane circumradius of ABC.

If we wish to maximize the distance from the origin to the closest facet of the convex hull of the points, we have just to find a polyhedron with N vertices on the sphere where the largest (spherical) circumcenter becomes as small as possible. If ρ denotes this largest (spherical) circumradius, we obtain a covering of the sphere using the following way:

Draw around each vertex of the polyhedron a circle of radius ρ. These circles overlap and each point of the sphere belongs to at least one circle.

If we find the smallest covering of the sphere by N equal circles for any N we get also the polyhedron with N vertices on the sphere with largest spherical circumcenter as small as possible and the arrangement of spherical points where the distance from the origin to the closest facet of the convex hull becomes as large as possible. Therefore the problems are equivalent. $\diamondsuit$

Literatur

[1] Fejes-Tóth,Gabor:
Kreisüberdeckungen der Sphäre
Stud. Sci. Math. Hung. **4** (1969), 225-247

[2] Fejes-Tóth,Lászlo:
Lagerungen in der Ebene, auf der Kugel und im Raum
Springer, Berlin-Heidelberg, 1956, 1972

[3] Schütte, Kurt:
Überdeckung der Kugel mit höchstens acht Kreisen
*Math. Ann.***129** (1955),181-186

[4] Tarnai,Tibor;Gáspár,Zsolt:
Covering the Sphere by Equal Circles on a Sphere and the Rigidity of its Graph.
Math. Proc. Camb. Phil. Soc. **110** (1991),71-90

[5] Wimmer,Lienhard
Covering the Sphere with Eight or Nine Equal Circles
preprint avaible at: www.lienhard-wimmer.de